Self-excited soliton motion

J. A. González
Departamento de Física, Universidad de Camagüey, Circunvalación Norte, Camagüey 74650, Cuba

L. E. Guerrero and A. Bellorín
Centro de Física, Instituto Venezolano de Investigaciones Científicas, Apartado 21827, Caracas 1020 A, Venezuela
(Received 14 December 1994; revised manuscript received 13 February 1996)

We present an extensive analytical and numerical study of the dynamics of kink solitons in Klein-Gordon systems with nonlinear damping. Particularly, the nonlinear damping could model the interaction of the solitons with an active medium. We analyze the existence and stability conditions of stationary states for the soliton. We present a different kind of bifurcation: a structure-breaking bifurcation. After this bifurcation the soliton enters a highly nonstationary state (*solitonic explosion*). We show the existence of self-sustained oscillations of solitons (*solitonic limit cycles*). Finally, we present chaotic motion of solitons similar to the Duffing–Van der Pol type. [S1063-651X(96)05707-8]

PACS number(s): 05.45.+b, 52.35.Sb, 52.35.Mw, 02.30.Jr

I. INTRODUCTION

The nonlinear Klein-Gordon-like equations model a wide variety of physical situations and have received a great deal of attention in recent years. The dynamics of Klein-Gordon solitons or kinks in the presence of external forces [1,2], impurities [3,4], or heat baths [5,6] represents real condensed matter systems and phenomena.

Self-excited oscillations are a fascinating feature of the nature; this phenomenon is exhibited by certain simple systems with one degree of freedom [7] as well as by coupled oscillators [7,8] and continuous systems [7,9]. Self-excited motion of solitons has been studied [4] for the Korteweg–de Vries and the nonlinear Schrödinger equations.

In this paper we study the self-sustained motion of the Klein-Gordon soliton [10–18] in an active medium in the presence of nonlinear damping that can pump energy into the system. We analyze the sufficient condition for the existence of a whole discrete spectrum of stationary velocities for the soliton (even in the absence of external forcing) as well as the sufficient condition for the existence of a different structure-breaking bifurcation. Furthermore, we introduce the soliton limit cycle and explore the modeling of chaotic motion for the active Klein-Gordon system.

Our paper is organized as follows. In Sec. II we present a description of our model. In Sec. III the existence of soliton stationary states is analyzed and the exact solution of a particular case is considered. In Sec. IV we discuss the differences between the structure-breaking bifurcation for active Klein-Gordon systems and a similar bifurcation for forced Klein-Gordon systems with linear damping. In Sec. V we present soliton limit cycles and soliton explosions in the presence of a spatially inhomogeneous driving force. In Sec. VI we present the chaotic behavior of the soliton obtained by pumping energy into the translational mode of the kink. Finally, in Sec. VII we summarize and discuss our results and also present some concluding remarks. In the Appendix we outline the numerical method.

II. SOLITONS IN ACTIVE MEDIA

We are interested in soliton dynamics sustained by the interaction with an active medium in the absence of a driving force. This situation can be modeled by a Klein-Gordon-like system with nonlinear damping

$$\phi_{xx}-\phi_{tt}-R(\phi,\phi_t)+G(\phi)=0, \tag{1}$$

where $G(\phi)=-dU(\phi)/d\phi$, $U(\phi)$ being the nonlinear potential, and $R(\phi,\phi_t)$ is the damping term, which can be a nonlinear function of ϕ and ϕ_t and will introduce negative damping effects that will give energy to the soliton. In order to ensure the existence of solitonic solutions [12] we assume that the potential $U(\phi)$ is an analytical function of ϕ and that it possesses at least two minima at points ϕ_1 and ϕ_3 and a maximum at the point ϕ_2, $\phi_1<\phi_2<\phi_3$. We also assume that there are no other extrema in the interval $\phi_1<\phi<\phi_3$, $U(\phi_1)=U(\phi_3)=0$, and $R(\phi,0)=0$.

Equation (1) can be realized by considering the lumped transmission-line circuit presented in Fig. 1(a). Figure 1(b)

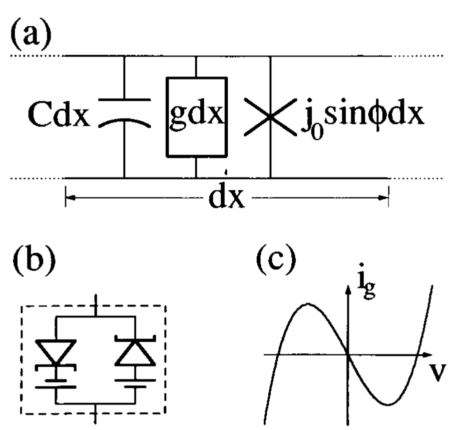

FIG. 1. Active transmission line. (a) Element of the lumped transmission-line circuit; $1/g$ is the nonlinear resistance. (b) Negative resistance twin-tunnel-diode circuit. (c) Nonlinear characteristic of the negative-resistance oscillator.

presents the negative-resistance twin-tunnel diode circuit; alternatively, a twin-transistor circuit can be employed [19]. Figure 1(c) shows the nonlinear driving-point characteristic that is described [19] by the function $i = gv$ (here $g = -B + Av^2$). Note that the element of the lumped transmission-line includes a Josephson junction (or its analog equivalent) whose supercurrent is described by two basic Josephson relations

$$j = j_0 \sin \phi, \tag{2a}$$

$$\frac{\partial \phi}{\partial T} \propto V, \tag{2b}$$

where V is the voltage and ϕ is the difference between the phases of the order parameters of the two superconductors layers of the junction. The equation of motion of the parameter ϕ is

$$\phi_{tt} - \phi_{xx} + a(\phi_t)^3 - b\phi_t + \sin \phi = 0, \tag{3}$$

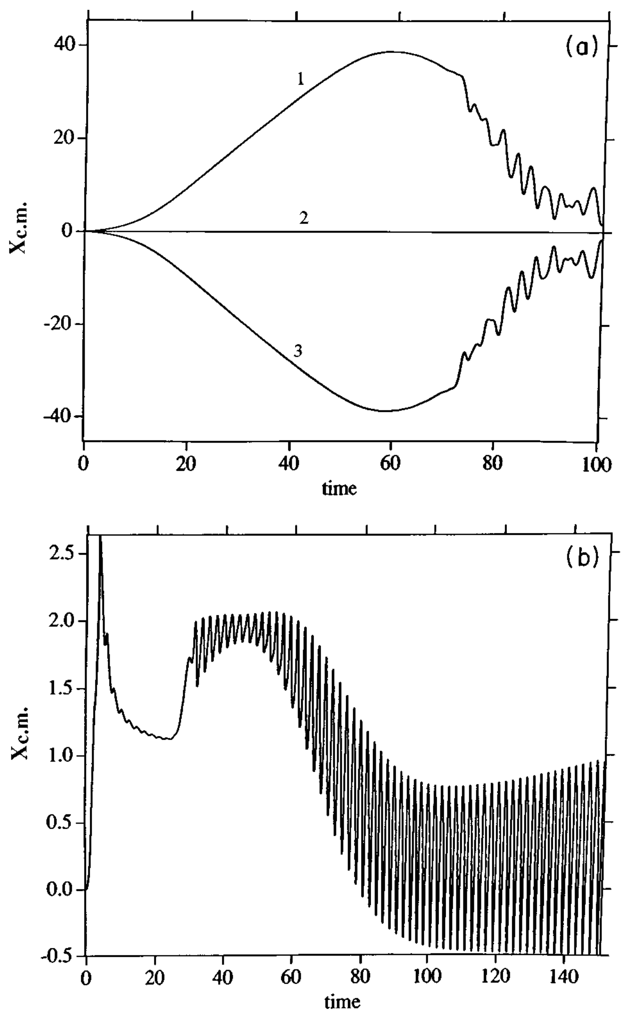

FIG. 2. ϕ^4 soliton in an active medium. Also shown is the temporal series for the center of mass of the soliton. (a) The kink reaches its stationary velocity before striking at the border of the system. Several initial velocities are considered. (b) The kink explodes very far from the edge of the system.

where time and space are measured in their natural units, so the equation is dimensionless (a and b are constants). Moreover, considering a coupled chain of Froude pendulums [20] leads to an expression like Eq. (3). Other nonlinear damping mechanisms are possible for the Josephson junction; for instance, assuming that the resistance varies with the voltage, a quadratic damping mechanism has been proposed [21,22].

III. SELF-SUSTAINED SOLITON MOTION

In this section we are concerned with the final dynamical state of a Klein-Gordon kink in the presence of nonlinear damping. For particular cases we give approximate and exact solutions.

A. Structure-breaking bifurcation

Solitons that move without change of shape and velocity [12,15,17,18] correspond to solutions of the equation

$$\phi_{zz} - R(\phi, -w\phi_z) + G(\phi) = 0 \tag{4}$$

(here $z = x - vt/\sqrt{1-v^2}$ and $w = v/\sqrt{1-v^2}$, v being the speed of the soliton) that satisfies the relation

$$\int_{-\infty}^{\infty} R(\phi, -w\phi_z)\phi_z dz = 0. \tag{5}$$

States of soliton movement with constant velocity and shape [10–18] are possible only if the equation

$$S(w) = -\int_{\phi_1}^{\phi_3} R[\phi, -w\sqrt{2U(\phi)}]d\phi = 0 \tag{6}$$

has a real solution with respect to w. Each real solution of Eq. (6) corresponds to a possible stationary state of the soliton. For small R, from Eq. (6) the velocities v of the solitons in these states can be calculated approximately.

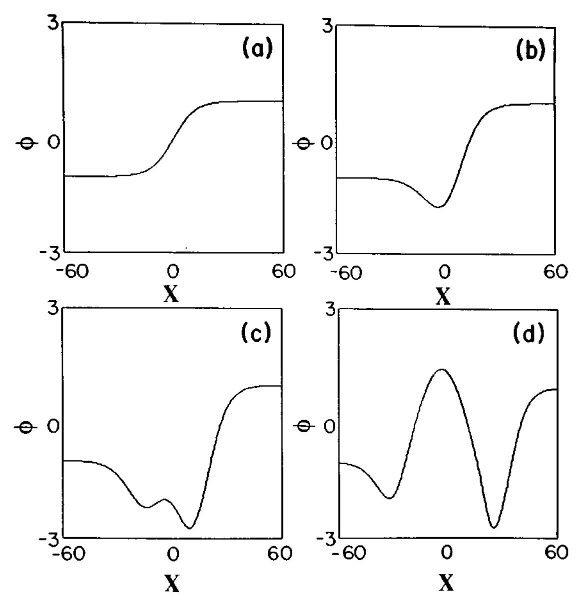

FIG. 3. (a)–(d) Soliton explosion. Spatial profiles [$\phi(x)$ vs x] corresponding to successive stages of the soliton destabilization.

If w_0 corresponds to a stationary velocity $v_{\rm stat}(w_0 = v_{\rm stat}/\sqrt{1-v_{\rm stat}^2})$ of the soliton, it is stable if

$$\left(\frac{\partial S(w)}{\partial w}\right)_{w=w_0} > 0. \tag{7}$$

This condition holds for the stability if $R(\phi,\phi_t)=R(-\phi,\phi_t)$. When the values of R are comparable with the absolute values of extrema of $G(\phi)$ in the interval $\phi_1<\phi<\phi_3$, a bifurcation can occur, after which the existence of the stationary state w_0 of the soliton is impossible. A sufficient condition for the bifurcation to have not occurred is that the inequality

$$U(\phi)+\int_{\phi_1}^{\phi} R[\phi,-w_0\sqrt{2U(\phi)}]d\phi>0 \tag{8}$$

holds in the interval $\phi_1<\phi<\phi_3$.

Let R_m and G_m be the absolute extreme values of $R[\phi, -w_0\sqrt{2U(\phi)}]$ and $G(\phi)$ in the interval $\phi_1<\phi<\phi_3$. Then the inequality

$$R_m > G_m \tag{9}$$

is a sufficient condition for the existence of the bifurcation. We would like to call this a *structure-breaking bifurcation*.

In an equation less general than Eq. (1),

$$\phi_{xx}-\phi_{tt}-R(\phi_t)+G(\phi)=0, \tag{10}$$

where $R(\phi_t)$ does not depend explicitly on ϕ, a necessary condition for the existence of stationary soliton states is that $R(\phi_t)$ has zeros. Each zero corresponds to a possible stationary state. Nevertheless, this condition is not sufficient.

Let us suppose that $R(\phi_t)$ is odd and has three zeros as in the case of Eq. (3). The state of rest of the soliton always exits. But the other two stationary states, in which the solitons have positive and negative constant velocities, are possible only if R_m is lower than a certain threshold value. When these two nontrivial velocities exist, they are stable. Meanwhile, the zero-velocity state is unstable.

Let us study the specific case of Eq. (3). For small values of b/a the stationary velocities of the solitons can be calculated by the formulas

$$w_{0_1}=0, \tag{11a}$$

$$w_{0_2}=\left(\frac{3}{2}\frac{b}{a}\right)^{1/2}, \tag{11b}$$

$$w_{0_3}=-\left(\frac{3}{2}\frac{b}{a}\right)^{1/2}, \tag{11c}$$

which are the roots of Eq. (6). According to Eq. (9), when

$$\frac{b^3}{a}>\frac{27}{4}, \tag{12}$$

the bifurcation has already occurred and there are no nontrivial stationary states.

The value of w corresponding to a stationary velocity of the soliton initially grows linearly with respect to $(b/a)^{1/2}$. Later the growth is steeper, and when b reaches the critical value, w tends to infinity. This means that the soliton reaches its highest possible velocity ($v=1$) for a finite value of b/a.

Alternatively, we can consider the ϕ^4-type equation

$$\phi_{xx}-\phi_{tt}+b\phi_t-a(\phi_t)^3+\tfrac{1}{2}(\phi-\phi^3)=0. \tag{13}$$

In order to avoid the structure-breaking bifurcation, the following condition must be verified:

$$2b\sqrt{b/a}<1. \tag{14}$$

Figure 2(a) presents the temporal evolution of kinks with several initial velocities ($A=1.0$, $B=0.5$, $a=0.2$, $b=0.2$, $x_0=0.0$, and length $l=120.0$); it can be noted that for the nonzero initial velocity cases the solitons acquires a constant velocity before striking the border of the system. After this collision, we have verified that there is a frustrated attempt to reflect an anti-kink; nevertheless, what emerges after the collision is again a kink. However, the reflected kink exhibits an active exchange of energy between the traslational mode and the internal modes (of the soliton) that precludes another engagement of a constant velocity. Figure 2(b) presents, for the same parameters but $b=2.0$, the explosion of the soliton when its center of mass is very far from the border of the system and during the initial stages of the motion. Figures 3(a)–3(d) presents snapshots of the evolution of the spatial profile in the case in which the soliton explodes, whereas Fig. 4 shows the temporal evolution before and after the soliton explosion.

B. Exact solution

In this section we present a case with an exact solution. Consider Eq. (1) in the particular case in which

$$R(\phi,\phi_t)=-b\phi_t+a(\phi_t)^3, \tag{15}$$

$$-G(\phi)=A_1\phi+A_3\phi^3+A_5\phi^5+A_7\phi^7+A_9\phi^9. \tag{16}$$

The stationary solitons for this case are solutions of the equation

$$\phi_{zz}+R(\phi_z)+G(\phi)=0, \tag{17}$$

where $R(\phi_Z)=-\delta\phi_z+\gamma(\phi_z)^3$, $\delta=bw$ and $\gamma=aw^3$.

Following the method developed by Otwinowski, Paul, and Laidlaw [23], it is possible to obtain exact solutions of Eq. (17) when the parameters of the system fulfill certain requirements. If

$$\phi_z=\varepsilon_1\phi+\varepsilon_3\phi^3, \tag{18}$$

where ε_1 and ε_3 are unknown parameters, then it is possible to integrate Eq. (17) in quadrature. For a solutions to exist satisfying Eq. (18), it is necessary that ε_1 and ε_3 satisfy the system of algebraic equations

$$A_1=\varepsilon_1^2-\delta\varepsilon_1, \tag{19a}$$

$$A_3=4\varepsilon_1\varepsilon_3-\delta\varepsilon_1^3-\delta\varepsilon_3, \tag{19b}$$

$$A_5=3\varepsilon_3^2+3\gamma\varepsilon_1^2\varepsilon_3, \tag{19c}$$

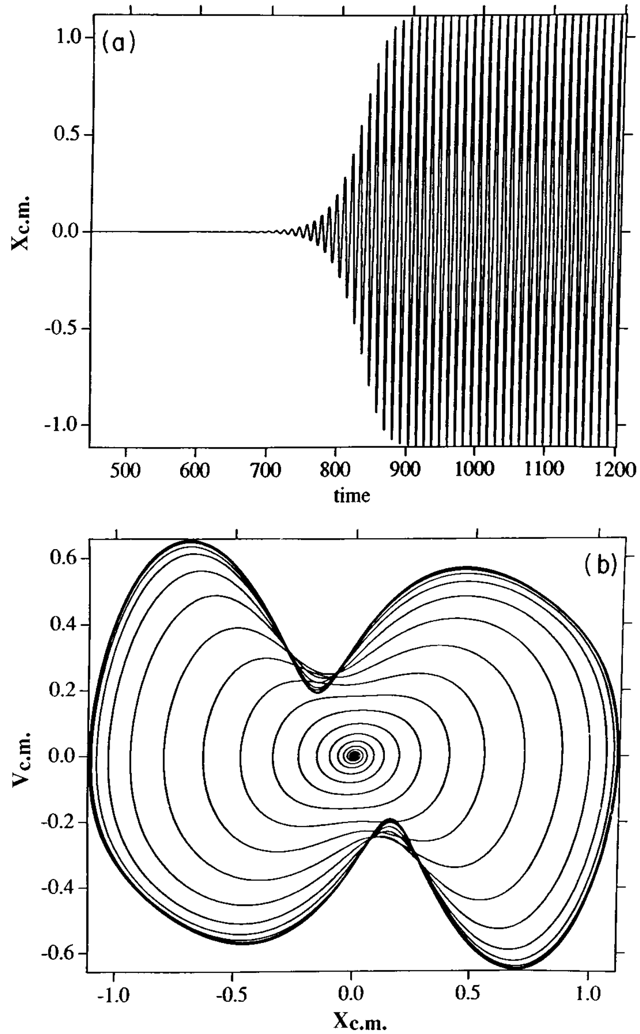

FIG. 5. Soliton limit cycle. (a) Time series $x_{\text{c.m.}}$ vs t for the position of the center of mass of the soliton in the absence of external forcing. (b) Phase space $v_{\text{c.m.}}$ vs $x_{\text{c.m.}}$ showing transient behavior towards a limit cycle.

$$A_7=3\gamma\varepsilon_1\varepsilon_3^2, \tag{19d}$$

$$A_9=\gamma\varepsilon_3^3. \tag{19e}$$

We have to make the adjustment of the parameters ε_1, ε_3, and w, so there are only three independent parameters A_i. For simplicity we choose as independent parameters A_5, A_7, and A_9. Then parameters ε_1 and ε_3 can be calculated by the formulas

$$\varepsilon_1^2=\frac{1}{9}\left(\frac{A_7}{A_9}\right)^2\left(\frac{A_5A_9-A_7^2}{3A_9}\right)^2, \tag{20a}$$

$$\varepsilon_3^2=\left(\frac{A_5A_9-A_7^2}{3A_9}\right)^2. \tag{20b}$$

We are interested in the kink soliton due to a connection between points $\phi_1=0$ and ϕ_3, which is the closest minimum of the potential $U(\phi)$ (there can be other minima). Actually, the potential is symmetrical around zero, so when there is a connection between $\phi_1=0$ and ϕ_3 there is also a connection between $\phi_1=0$ and $-\phi_3$. In fact,

$$\phi_3=\pm\left(-\frac{\varepsilon_1}{\varepsilon_3}\right)^{1/2}=\pm\left(-\frac{A_7}{3A_9}\right)^{1/2}. \tag{21}$$

It is now evident that these parameters must satisfy the following relationships for the soliton to exist:

$$\varepsilon_1\varepsilon_2<0, \tag{22a}$$

$$A_7A_9<0, \tag{22b}$$

$$A_9(A_9A_5-A_7^2)<0. \tag{22c}$$

On the other hand, if we require equal height of the minima of $U(\phi)$,

$$U(\phi_1)=U(\phi_3)=0, \tag{23}$$

then w is solution of

$$bw+Paw^3=0, \tag{24a}$$

$$P=\frac{A_7^3}{270A_9^3}\left(\frac{A_5A_9-A_7^2}{3A_9}\right). \tag{24b}$$

The parameters A_1 and A_3 will appear as functions of the rest of the parameters through Eqs. (19a), (19b), (20), and (24). The exact solution is

$$\phi=\frac{\phi_3}{\sqrt{1+e^{-2\varepsilon_1 z}}}. \tag{25}$$

Note that the soliton can have three different stationary velocities that are solutions of Eq. (24). However, there is a condition for parameters a and b,

$$\frac{b^3}{a}<\frac{A_7^3}{729A_9^3}\,(A_5A_9-A_7^2), \tag{26}$$

which is equivalent to the restrictions imposed above [Eqs. (9) and (12)] on a and b. We stress the fact that given fixed A_5, A_7, and A_9, for large values of $b\sqrt{b/a}$ there is no stationary soliton.

IV. LINEAR VS NONLINEAR DAMPING

It is interesting to compare the type of bifurcation that we have uncovered with a different kind of bifurcation that does not allow the existence of stationary solitons in systems with an external perturbation [14,24]. The most simple case is

$$\phi_{xx}-\phi_{tt}-\gamma\phi_t+\tfrac{1}{2}(\phi-\phi^3)=-F. \tag{27}$$

when $F^2<\frac{1}{27}$ there is a stationary velocity for the soliton, but when $F^2>\frac{1}{27}$ the stationary soliton is not longer possible.

For the sine-Gordon-like equation

$$\phi_{tt}-\phi_{xx}-\gamma\phi_t+\sin\,\phi=-F \tag{28}$$

the critical value is $F^2=1$. The explanation of this phenomenon is that a system described by Eq. (1) must have at least

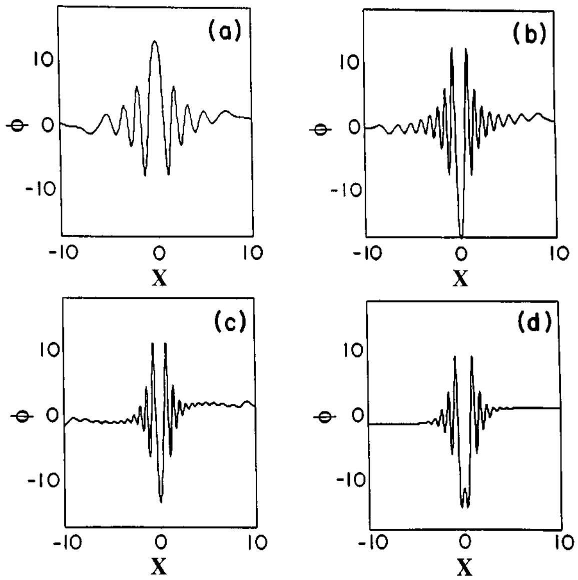

FIG. 6. (a)–(d) Soliton explosion. Spatial profiles [$\phi(x)$ vs x] corresponding to the final stages of the soliton destabilization.

three fixed points (two of them stable) for solitons to be able to exist [20,14–18,23]. When $F^2 > \frac{1}{27}$ two of the fixed points of Eq. (27) disappear and therefore the solitons cannot exist. In contrast, even when the condition expressed by Eq. (9) is fulfilled, the system [Eq. (10)] has all its fixed points in their place. Given Eq. (9), the perturbation can destabilize the soliton without destroying the stable phases of the system. Other results [12,15,24] lead to the idea that the internal structure of the soliton cannot resist in a stable way any external force greater than a certain critical value, even when the fixed points of the system are conserved.

In systems of the kind described by Eq. (3), as well as in systems of the type given by Eqs. (27) and (28), there is damping and pumping at the same time. Common sense tells us to expect some compromise between these opposing actions at a certain velocity (this is the case before the bifurcation). But under certain conditions, this compromise is impossible. This may be a universal phenomenon in extended spatiotemporal structures with internal dynamics. While the soliton is in one of the stationary states it preserves its shape and velocity without changes.

We have to stress that in the case given by Eq. (3) the fixed points are locally unstable due to the negative damping in their neighborhood. So, after perturbations the tails of the soliton will perform self-sustained small oscillations. But the soliton as a whole will move with almost constant velocity. We must stress also the point that unlike systems with linear damping, in our system the soliton can have a whole discrete spectrum of stationary velocities.

V. SOLITON LIMIT CYCLES

Historically, linear equations were used to describe self-excited electrical oscillations. Only the introduction of limit cycles in the theory of electrical oscillations made possible the construction of models describing all the properties of the phenomena [25].

Like the harmonic oscillator, the classical integrable equations that exhibit solitons (Korteweg–De Vries, sine-Gordon, etc.) are structurally unstable. The real quasistationary solitons must be described by models that include spatiotemporal attractors as self-sustained solitons.

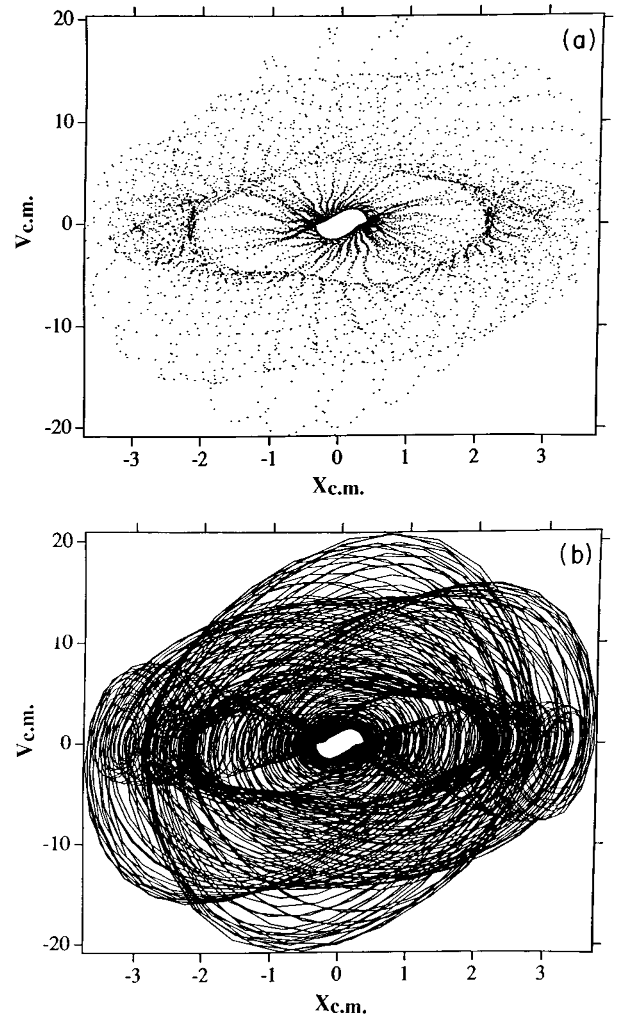

FIG. 7. Fractal supertorus. (a) Poincaré map for the temporal evolution of the center of mass of the soliton. (b) Corresponding phase plane.

Recent studies [14,24] of the dynamics of solitons under the action of external forces have shown that equations of the type

$$\phi_{xx} - \phi_{tt} - \gamma\phi_t + G(\phi) = -F(x) \tag{29}$$

have solutions that describe solitons whose centers of mass perform damped oscillations around certain stable equilibrium positions that are located in certain zeros x_0 of $F(x)$ [$F(x_0)=0$]. Now we can state that if the system, in addition to an external nonhomogeneous force $F(x)$ (with a stable zero), has nonlinear damping,

$$\phi_{xx} - \phi_{tt} - R(\phi,\phi_t) + G(\phi) = -F(x), \tag{30}$$

then the center of mass of the soliton can perform self-sustained oscillations: we are in the presence of a soliton limit cycle. An example of an equation with this property is

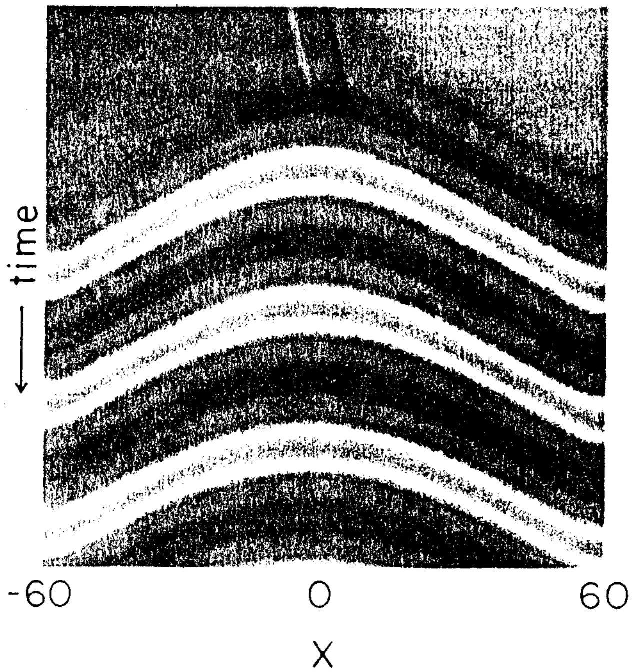

FIG. 4. Soliton explosion. Temporal evolution of the solution. Areas depicted with blue correspond to lower values of the profile, whereas yellow and red areas denote higher values. The kink profile therefore can be identified at earlier times.

$$\phi_{xx}-\phi_{tt}+b\phi_t-a(\phi_t)^3+\tfrac{1}{2}(\phi-\phi^3)=B(1-4B^2)\tanh(Bx), \tag{31}$$

where $4B^2>1$. Here we have used the inhomogeneous static

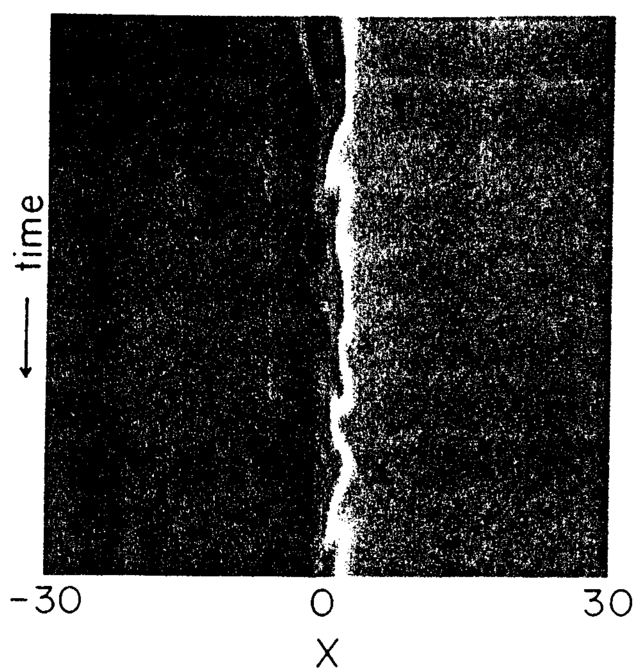

FIG. 8. Spatiotemporal evolution corresponding to the fractal supertorus attractor. Violet and blue denote lower values of the solution $\phi(x,t)$, whereas yellow and red depict higher values of the solution.

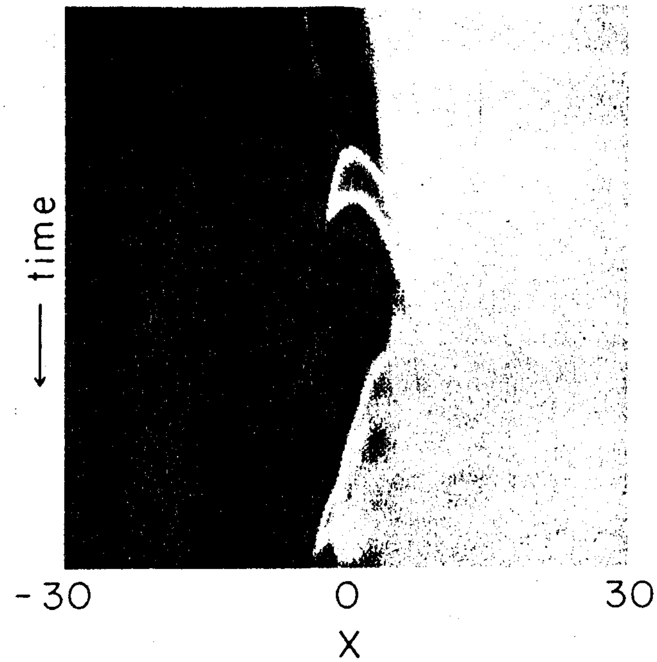

FIG. 10. High-dimensional chaotic motion of one soliton. Spatiotemporal evolution of the soliton. Blue and green denote lower values of the solution $\phi(x,t)$, whereas yellow and red correspond to higher values.

force introduced in Ref. [24] for which the exact stability problem has been solved. In order to avoid the bifurcation described above, we need the fulfillment of the condition (14).

Another kind of soliton limit cycle can be found in equations of the type

$$\phi_{xx}-\phi_{tt}-\Gamma(x)\phi_t+G(\phi)=-F(x), \tag{32}$$

where the damping coefficient is a function of x and has intervals of negative and positive values. For instance,

$$\phi_{xx}-\phi_{tt}-\Gamma(x)\phi_t+\tfrac{1}{2}(\phi-\phi^3)=B(1-4B^2)\tanh(Bx), \tag{33}$$

where $\Gamma(x)=\gamma[1-L/\cosh^2(Dx)]$, $1-L<0$.

Figure 5(a) presents the temporal series for the onset of self-sustained oscillations of the center of mass of the soliton, whereas Fig. 5(b) presents the corresponding phase-space portrait ($A=1.3$, $B=0.65$, $\gamma=0.15$, $L=2.0$, $D=0.65$, $x_0=0.0$, $v_0=0.0$, and length $l=20.0$). The solution evolves out from an unstable focus towards a limit cycle. We remark that the solution corresponds to the periodic motion of a soliton.

It is interesting that when

$$\gamma(L-1)>\frac{1}{3\sqrt{3}} \tag{34}$$

we will observe a soliton explosion due to the structure-breaking bifurcation as shown in Fig. 6, where we present a sequence of snapshots of the spatial profile $\phi(x)$ vs x revealing loss of the kink topology (all parameters are the same as

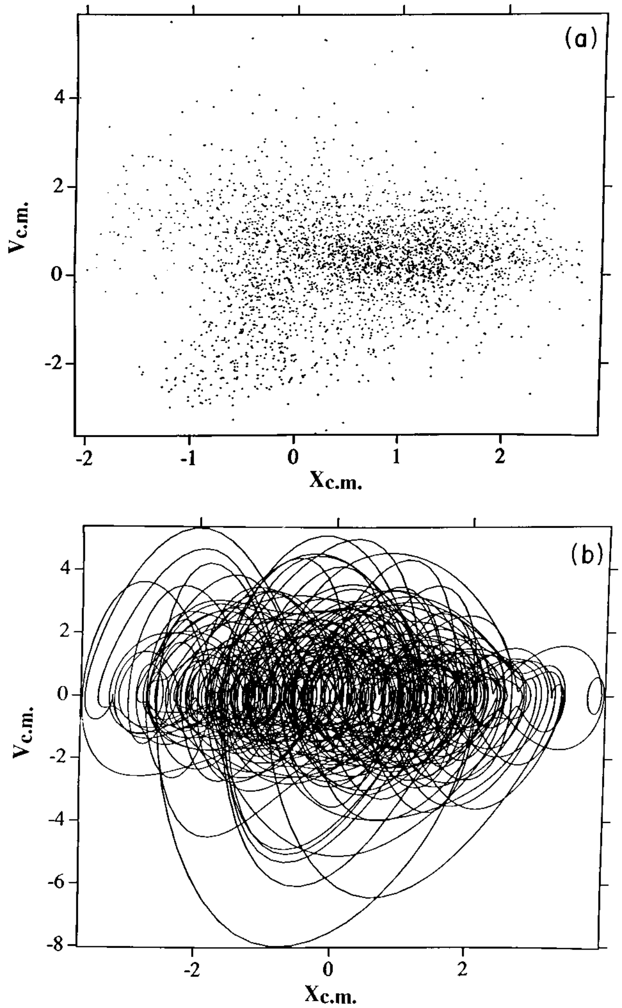

FIG. 9. (a) Poincaré map revealing the high-dimensional chaotic motion of one soliton. (b) Corresponding phase space $v_{\mathrm{c.m.}}$ vs $x_{\mathrm{c.m.}}$.

in the previous case except $\gamma=0.25$ and $L=6.0$), i.e., the soliton used as initial condition has been destabilized due to the constraints and conditions imposed on the system. Time series also reveal that the position and the velocity of the center of mass of the solution are practically zero at all times except while the soliton explodes.

VI. CHAOTIC SOLITONS

We are concerned with controlling the dynamic behavior of solitons: this can be achieved with the knowledge of the effect of every term that is introduced in the equations. In this respect, the chaotic behavior of solitons has been previously *predicted* [15,24] and verified [26] in equations like Eq. (29) when $F(x)$ has two stable zeros and there is an additional time-periodic force acting on the system. In such a case the behavior of the soliton is similar to the motion of a particle in Duffing's equation with two stable equilibrium positions.

Consider the following system, which represents an equation of the type given by Eq. (32) perturbed by a time-periodic force:

$$\phi_{xx}-\phi_{tt}-\Gamma(x)\phi_t+\tfrac{1}{2}(\phi-\phi^3)=\tfrac{1}{2}A(1-A^2)\tanh^3(Bx)+g\,\frac{\cos(\Omega_d t)}{\cosh^2(Bx)}, \tag{35}$$

where $\Gamma(x)$ is defined as in Eq. (33). Here we have introduced an inhomogeneous static forcing proportional to x^3 near zero in order to have a Duffing-like force (this is obtained when $B=\frac{1}{2}$ in the inhomogeneous static force introduced in Ref. [24]). The particular time-dependent force selected pumps energy only into the translational mode of the kink; this is due to the fact that this force is fitted to the shape of the translational mode [24].

For some set of parameters Eq. (35) will have complex dynamics of the center of mass of the soliton similar to the behavior of ac-driven Duffing–Van der Pol equation [27]. In addition to the natural frequency of the system (the one that arises from the soliton limit cycle) there is the frequency of the time-dependent force. Quasiperiodic behavior can be excited if these frequencies are incommensurable. Notwithstanding, even richer dynamical behavior can also be exhibited by the system. Figure 7(a) presents the Poincaré map for the center of mass of the soliton, which reveals a *fractal supertorus* ($A=3.0$, $B=0.5$, $\gamma=0.1$, $L=3.0$, $D=0.6$, $g=1.0$, $\Omega_d=0.65$, $x_0=0.0$, $v_0=0.1$, and length $l=60.0$). It can be appreciated that the destruction of a central torus (it is no longer a closed curve) has given rise to a new and larger ramified structure; notice the dendritic patterns. Figure 7(b) presents the corresponding phase plane; the forbidden central region is due to the presence of an unstable focus (the same involved in the creation of the soliton limit cycle that becomes a quasiperiodic attractor when the time-dependent force is introduced). Finally, spiral escaping orbits corresponding to unstable trajectories can be appreciated.

We stress that this strange attractor corresponds to chaotic motion of a kink soliton around its equilibrium position as presented in Fig. 8. We have verified that the soliton preserves its structure despite the chaotic appearance and disappearance of minor protuberances due to the excitation of

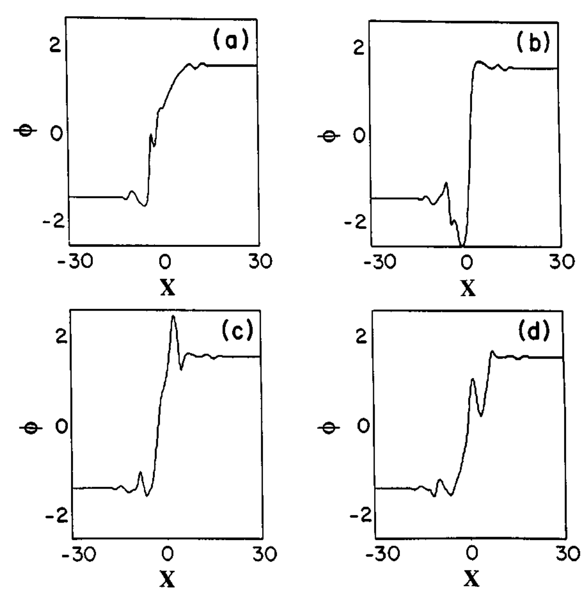

FIG. 11. (a)–(d) High-dimensional chaotic motion of the soliton. Snapshots of the solution.

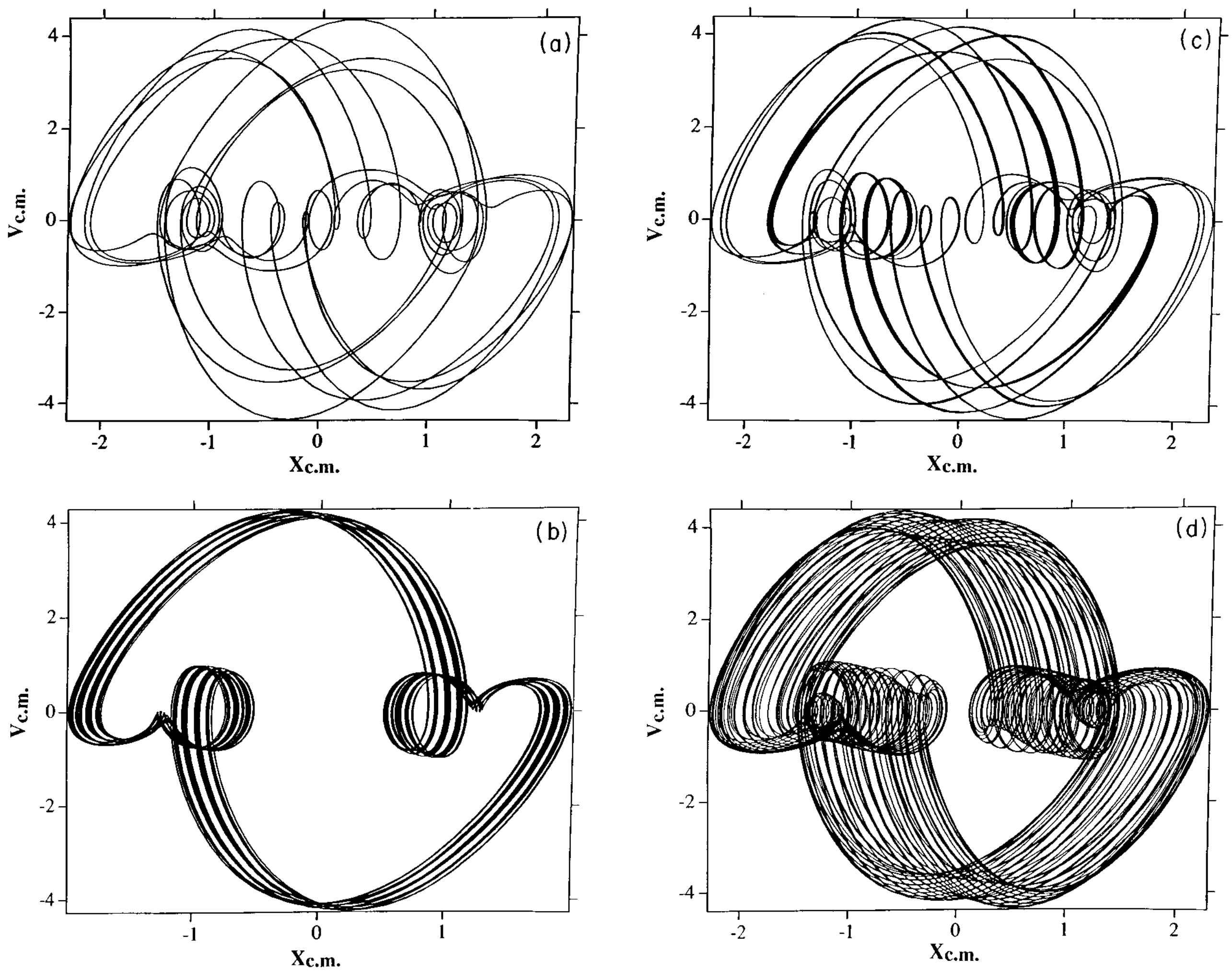

FIG. 12. ''Duffing-like'' dynamics of the soliton. (a) Phase plane corresponding to a high-period oscillation of the soliton (γ=0.1, L=0.0). (b)–(d) Solutions for γ=0.118, 0.109, and 0.106, respectively.

shape modes [24]. The chaos in the shape coexists with the chaotic motion of the soliton. However, we emphasize that the chaos in form plays a minor role in the case of the fractal supertorus.

Decreasing $|A|$ leads to a destabilization of the soliton motion [24]. For $A=1.5$ (the rest parameters are the same as the previous case) the Poincaré map [Fig. 9(a)] reveals an activation of a larger number of effective degrees of freedom, which increases the dimension of the attractor and results in the complete destruction of the fractal supertorus attractor. Since we are monitoring, in our plots, the dynamics of the center of mass, the high dimension of these Poincaré maps can be ascribed to symmetry breaking of the profile associated with the excitation of the internal modes. Figure 10 presents the temporal evolution of the soliton, whereas Figs. 11(a)–11(d) show snapshots of the corresponding profiles revealing an increased excitation of shape modes while the kink profile is still sustained.

In order to clarify the mechanisms that are involved in the appearance of the attractor of Fig. 9 we set $L=0$: this amounts to ceasing the pumping of energy and allows the unveiling of the underlying ''Duffing-like'' dynamics of the soliton. Figure 12(a) presents the phase plane for $A=1.5$ and $L=0$ (the rest parameters remain unchanged); this phase plane corresponds to the dynamics of the center of mass and shows a high-period solution in contrast with the chaotic motion presented in Fig. 9(b) for $L=3.0$. Notice that the internal modes are no longer excited. Diminishing γ in the interval γ=0.125 to 0.096 leads the oscillations of the soliton from period one to chaos through a rich sequence of bifurcations as shown in Figs. 12(b) and 12(c) for three selected solutions at γ=0.118, 0.109, and 0.106.

The underlying mechanism of the fractal supertorus (Fig. 7) then becomes clear. This attractor results from the combination of two types of chaos: the Duffing-like chaos and the chaos resulting from the time-periodic perturbation of the (self-sustained) limit cycle. Decreasing the parameter A results in the destruction of the fractal supertorus since the Duffing-like chaos then leads the dynamics.

VII. SUMMARY AND CONCLUSIONS

In this paper we have presented the self-sustained soliton as an interesting paradigm of real solitons. We have shown that the oscillation of a soliton exhibits a surprising richness and brings different spatiotemporal phenomena.

We have predicted the destruction or the preservation of a kink profile by an active Klein-Gordon system. We stress that our analytical results made us able to predict the periodic, quasiperiodic, and chaotic motion of a soliton as well as some of the general features of such chaotic motion. We have also presented an alternative spatiotemporal dynamics in which two types of chaos collide while they describe the oscillation of the center of mass of a soliton.

The soliton limit cycle might provide a versatile source of radiation. The discrete spectrum of velocities for the soliton also presents practical importance as it allows the soliton to jump between several states.

ACKNOWLEDGMENTS

This work has been partially supported by Consejo Nacional de Investigaciones Científicas y Tecnológicas under Project No. S1-2708. J.A.G. gratefully acknowledges the hospitality of the Low Temperatures Laboratory at the Instituto Venezolano de Investigaciones Científicas. L.E.G. gratefully acknowledges the hospitality of the Nonlinear Dynamics Systems Group at the University of Camagüey.

APPENDIX

We have integrated our ϕ^4-like equations using a standard implicit finite-difference method. In this paper we employed open boundary conditions $\phi_x(0,t)=\phi_x(l,t)=0$. We have used 0.035 for the time step and 0.039 for the space step.

Our numerical integrations were started using a kink soliton as the initial condition:

$$\phi(x,0)=A\tanh\left[\frac{B(x-x_0)}{\sqrt{1-v_0^2}}\right], \tag{A1}$$

$$\phi_t(x,0)=\frac{-ABv_0}{\sqrt{1-v_0^2}}\cosh^2\left[\frac{B(x-x_0)}{\sqrt{1-v_0^2}}\right], \tag{A2}$$

where A and B are constants and x_0 and v_0 are, respectively, the initial position and velocity of the center of mass of the soliton. We have defined the position of the center of mass of the kink soliton as

$$x_{\text{c.m.}}=\frac{\int_{-l/2}^{l/2}x(\phi_x)^2dx}{\int_{-l/2}^{l/2}(\phi_x)^2dx}, \tag{A3}$$

where l is the length of the system.

[1] St. Pnevmatikos, N. Flytzanis, and A. R. Bishop, J. Phys. C. **20**, 2829 (1987).
[2] D. W. McLaughlin and A. C. Scott, Phys. Rev. A **18**, 1652 (1978).
[3] T. Fraggis, St. Pnevmatikos, and E. N. Economou, Phys. Lett. A **142**, 361 (1989).
[4] F. Abdullaev, *Theory of Solitons in Inhomogeneous Media* (Wiley, Chichester, 1994).
[5] L. E. Guerrero, A. Hasmy, and G. J. Mata, Physica B **194-196**, 1631 (1994); R. Rangel, L. E. Guerrero, and A. Hasmy, *ibid.* **194-196**, 411 (1994).
[6] V. V. Konotop and L. Vázquez, *Nonlinear Random Waves* (World Scientific, Singapore, 1994).
[7] A. V. Gaponov-Grekhov and M. I. Rabinovich, *Nonlinearities in Action* (Springer-Verlag, Hong Kong, 1992).
[8] G. V. Osipov and M. M. Sushchik, Phys. Lett. A **201**, 205 (1995).
[9] A. A. Borissov and O. V. Sharypov, J. Fluid Mech. **257**, 451 (1993).
[10] K. Lonngren and A. Scott, *Solitons in Action* (Academic, New York, 1978).
[11] A. R. Bishop, J. A. Krumhansl, and S. E. Trullinger, Physica D **1**, 1 (1980).
[12] J. A. González and J. A. Holyst, Phys. Rev. B **35**, 3643 (1987).
[13] J. A. González and J. Estrada-Sarlabous, Phys. Lett. A **140**, 189 (1989).
[14] Y. S. Kivshar and B. A. Malomed, Rev. Mod. Phys **61**, 763 (1989).
[15] J. Estrada-Sarlabous and J. A. González, Academy of Sciences of Cuba, ICIMAF Research Report No. 1, p. 1, 1989 (unpublished).
[16] J. A. González, Mod. Phys. Lett. B **6**, 1867 (1992).
[17] J. A. González, Rev. Mex. Fis. **38**, 205 (1992).
[18] J. A. González, International Centre for Theoretical Physics Report No. IC/93/100, 1993 (unpublished).
[19] L. O. Chua, C. A. Desoer, and E. S. Kuh, *Linear and Nonlinear Circuits* (McGraw-Hill, New York, 1987).
[20] N. Minorsky, *Nonlinear Oscillations* (Van Nostrand, Princeton, 1962).
[21] N. F. Pedersen and K. Saermark, Physica **69**, 572 (1973).
[22] M. Bartuccelli, P. L. Christiansen, N. F. Pedersen, and M. P. Soerensen, Phys. Rev. B **33**, 4686 (1986).
[23] M. Otwinowski, R. Paul, and W. G. Laidlaw, Phys. Lett. A **128**, 483 (1988).
[24] J. A. González and J. A. Holyst, Phys. Rev. B **45**, 10 338 (1992).
[25] A. A. Andronov, E. A. Vitt, and S. E. Khaiken, *Theory of Oscillators* (Pergamon, Oxford, 1966).
[26] J. A. González *et al.* (unpublished).
[27] J. Guckenheimer and P. L. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields* (Springer-Verlag, Berlin, 1986).