Arbitrarily large numbers of kink internal modes in inhomogeneous sine-Gordon equations

J.A. González [a,b], A. Bellorín [c], M.A. García-Ñustes [d], L.E. Guerrero [e,*], S. Jiménez [f], L. Vázquez [g]

[a] *Department of Physics, Florida International University, Miami, FL 33199, USA*
[b] *Department of Natural Sciences, Miami Dade College, 627 SW 27th Ave., Miami, FL 33135, USA*
[c] *Escuela de Física, Facultad de Ciencias, Universidad Central de Venezuela, Apartado Postal 47586, Caracas 1041-A, Venezuela*
[d] *Instituto de Física, Pontificia Universidad Católica de Valparaíso, Casilla 4059, Chile*
[e] *Departamento de Física, Universidad Simón Bolívar, Apartado Postal 89000, Caracas 1080-A, Venezuela*
[f] *Departamento de Matemática Aplicada a las TT.II., E.T.S.I. Telecomunicación, Universidad Politécnica de Madrid, 28040-Madrid, Spain*
[g] *Departamento de Matemática Aplicada, Facultad de Informática, Universidad Complutense de Madrid, 28040-Madrid, Spain*

A R T I C L E   I N F O

A B S T R A C T

We prove analytically the existence of an infinite number of internal (shape) modes of sine-Gordon solitons in the presence of some inhomogeneous long-range forces, provided some conditions are satisfied.

© 2017 Elsevier B.V. All rights reserved.

1. Introduction

Solitary waves play a crucial role in many physical phenomena [1–57]. Recently there has been much discussion about the existence of internal (shape) modes of solitary waves [10–45], and several novel physical effects have been discovered in nonintegrable models possessing internal (shape) modes [28–57].

The sine-Gordon equation, which is an integrable partial differential equation that does not have internal (shape) modes, is one of the paradigms in soliton theory. This is a fundamental model in many areas of physics. For instance, it describes dislocations in crystals and fluxons in long Josephson junctions, just to mention two examples [6–9].

Integrable models describe real physical systems only with certain approximation [14]. So it is very important to study the sine-Gordon equation in the presence of perturbations. Many papers have been dedicated to the study of periodic time-dependent perturbations in the sine-Gordon model [15–22]. The possible observation of a "quasimode" in this model has been debated intensively in literature [15–27]. Some of these works [22,27] affirm that the mentioned "quasimode" is a numerical artifact.

In Ref. [23], the authors have shown the existence of an internal (shape) mode of the sine-Gordon kink when it is in the presence of inhomogeneous space-dependent external forces, provided that these forces possess some particular properties.

In the present letter we will investigate the inhomogeneous sine-Gordon equation

$$\phi_{tt} - \phi_{xx} + \sin\phi = F(x), \qquad (1)$$

where the external perturbation $F(x)$ decays as a power-law.

We will prove the existence of an infinite number of shape modes of sine-Gordon kinks in the presence of some long-range forces.

The paper is organized as follows. In Section 2, we introduce some important concepts for the understanding of the paper. In Section 3, we discuss the sine-Gordon equation perturbed by an exponentially-decaying force. In Section 4, we analyze the behavior of the kink in the presence of perturbations with a power-law decay. Finally, in Section 5, we present the conclusions of our analysis.

---

* Corresponding author.
  E-mail addresses: jalbertgonz@yahoo.es (J.A. González), alberto.bellorin@ucv.ve (A. Bellorín), monica.garcia@pucv.cl (M.A. García-Ñustes), lguerre@usb.ve (L.E. Guerrero), s.jimenez@upm.es (S. Jiménez), lvazquez@fdi.ucm.es (L. Vázquez).

2. Internal (shape) modes

Before discussing the new results, we would like to clarify some important concepts.

Consider the general equation

$$\phi_{tt} - \phi_{xx} + G(\phi) = F(x). \tag{2}$$

Suppose this equation supports a kink solution $\phi_k(x)$ [10–12,14, 15,28,58–62].

To investigate the stability of the kink [10–12,14,15,28,58–62], we look for solutions in the form $\phi(x,t) = \phi_k(x) + f(x)e^{\lambda t}$. The perturbation $f(x)$ must satisfy a Schrödinger-like equation

$$[-\partial_{xx} + U(x)] f(x) = \Gamma f(x), \tag{3}$$

where $U(x) = \partial G(\phi_k(x))/\partial\phi$, $\Gamma = -\lambda^2$.

In general, Eq. (3) has a discrete spectrum and a continuous spectrum [10–12,14,15,28,58–62].

The soliton modes are described by the eigenfunctions corresponding to the discrete spectrum [31]. On the other hand, the eigenfunctions corresponding to the continuous spectrum are generally called the phonon modes [10–12,14,15,28,58–62]. So the bound states of Eq. (3) are the soliton modes.

The soliton modes and the phonon modes determine the form of the oscillations around the kink solution.

Let us discuss the ϕ^4 equation as an example:

$$\phi_{tt} - \phi_{xx} - \phi + \phi^3 = 0. \tag{4}$$

The static kink solution is $\phi_k(x) = \tanh(x/\sqrt{2})$.

The equation for $f(x)$ is

$$-\partial_{xx} f(x) + \left[2 - 3\,\mathrm{sech}^2(x/\sqrt{2})\right] f(x) = \Gamma f(x). \tag{5}$$

The eigenvalues and eigenfunctions are given by the well-known expressions: for $\Gamma_0 = 0$ we get $f_0(x) = \mathrm{sech}^2(x/\sqrt{2})$, for $\Gamma_1 = 3/2$ we obtain $f_1(x) = \tanh(x/\sqrt{2})\mathrm{sech}(x/\sqrt{2})$, and $\Gamma_k = 2 + k^2$ correspond to the continuum spectrum. These are the (unnormalized) wave functions.

These solutions correspond to the translation mode ($f_0(x)$), the internal (shape) mode ($f_1(x)$), and the continuum phonons scattered by the kink [10–12,14,15,28,58–62]. In this case, the translation mode and the internal (shape) mode are the soliton modes.

The internal (shape) mode is responsible mostly for the vibrations of the kink width. The internal (shape) mode, with frequency $\omega = \sqrt{\Gamma} = \sqrt{3/2}$, represents a localized deformation around the kink and describes oscillations of the kink shape. This mode has been used to explain the complex resonance phenomena that occur during kink–antikink collisions [10–12,14,15,28,58–62]. We should remark that there are systems with several internal (shape) modes [11,23,28,63].

The continuum corresponds physically to dispersive travelling waves that propagate to spatial infinity [11].

Another very important example is the pure sine-Gordon equation

$$\phi_{tt} - \phi_{xx} + \sin\phi = 0. \tag{6}$$

Various solutions are known [10–12,14,15,28,58–62]. For instance, the static kink is

$$\phi_k(x) = 4\arctan[\exp(x)]. \tag{7}$$

In order to study the oscillation modes around the kink, we look for solutions in the form $\phi(x,t) = \phi_k(x) + f(x)e^{\lambda t}$, where $f(x)$ satisfies the equation

$$-\partial_{xx} f(x) + \left(1 - 2\,\mathrm{sech}^2(x)\right) f(x) = \Gamma f(x). \tag{8}$$

Eq. (8) has exactly only one bound eigenstate with $\Gamma = 0$ and

$$f_0(x) = \mathrm{sech}(x). \tag{9}$$

The zero-frequency bound state is the Goldstone [58] mode (also called the translation mode). The remaining eigenfunctions form a continuum with $\Gamma_k = 1 + k^2$. There are no internal (shape) modes in this case, there is only one soliton mode.

The birth of new internal (shape) modes have been discussed in several papers [14,23,28,63]. An appropriate perturbation can create a soliton internal (shape) mode [10–12,14,15,28,58–62].

The existence of several internal (shape) modes in the perturbed ϕ^4 equation is studied in Ref. [28] and the creation of several internal (shape) modes in the perturbed sine-Gordon equation is studied in Refs. [23,63]. The birth of an internal (shape) mode in the double sine-Gordon equation is presented in Ref. [14].

These phenomena can occur only when a new eigenvalue of the discrete spectrum is created [14].

3. Exponentially-decaying perturbation

The model

$$\phi_{tt} - \phi_{xx} + \sin\phi = F_1(x), \tag{10}$$

where $F_1(x) = 2(B^2 - 1)\sinh(Bx)/\cosh^2(Bx)$, was introduced in Ref. [23]. We recall here briefly the results presented in that work.

This force creates an equilibrium position for the center of mass of the kink. The equilibrated kink solution is $\phi_k = 4\arctan[\exp(Bx)]$. The stability investigation of this solution $\left[\phi(x,t) = \phi_k(x) + f(x)e^{\lambda t}\right]$ can be expressed as the eigenvalue problem $\widehat{L}f = \Gamma f$, where $\widehat{L} = -\partial_x^2 + \left(1 - 2\cosh^{-2}(Bx)\right)$ and $\Gamma = -\lambda^2$. This eigenvalue problem can be solved exactly [23,28,64]. The eigenvalues of the discrete spectrum can be calculated

$$\Gamma_n = B^2\left(\Lambda + 2\Lambda n - n^2\right) - 1, \tag{11}$$

where $\Lambda = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2}{B^2}}$ [23,28,64]. The smallest value of n is zero. The largest value of n is $n_{\max} = [\Lambda] - 1$, where $[\Lambda]$ is the integer part of Λ. The number of discrete eigenvalues is equal to the integer part of Λ.

If $1/3 < B^2 \le 1$, the integer part of Λ is one ($[\Lambda] = 1$). So there is only one soliton mode. In the case $B^2 = 1$, the Eq. (10) possesses translational invariance, and this state is called translational mode.

If $1/6 < B^2 \le 1/3$, then $[\Lambda] = 2$. So a new internal (shape) mode is created. If we continue decreasing parameter B, new eigenvalues of the discrete spectrum are created. All the new modes are internal (shape) modes.

4. Perturbation with a power-law decay

Consider now the completely new model

$$\phi_{tt} - \phi_{xx} + \sin\phi = F_2(x), \tag{12}$$

where $F_2(x) = 2B\left[(2B^2 - 1)x - B^2x^3\right]/\left(1 + B^2x^2\right)^2$.

Depending on parameter B, the perturbation $F_2(x)$ can have one or three zeroes.

For $(2B^2 - 1) > 0$, the function $F_2(x)$ has three zeroes. Otherwise, the only zero is $x = 0$.

We can find an exact solution for the stationary kink whose center of mass is at point $x = 0$:

$$\phi_k = 2\arctan(Bx) + \pi. \tag{13}$$

As $\partial\phi_k/\partial x = 2B/\left[1 + (Bx)^2\right]$, it is easy to see that for large values of $|x|$ the solution behaves as a power law. For instance, as $x \to \infty$, $(\phi_k - 2\pi) \sim 1/x$.

Compare this with the kink solution of Eq. (10) that possesses an exponential asymptotic behavior $(\phi_k - 2\pi) \sim e^{-Bx}$. Note that parameter B determines how fast the solution approaches the value 2π for large values of x. On the other hand, for $B \to 0$, the integer part of Λ ($[\Lambda]$) (which gives the number of soliton modes) behaves as $[\Lambda] \sim 1/B$. Thus the asymptotic behavior of the kink solution is related to the number of internal (shape) modes.

The stability analysis of solution (13) leads to the following eigenvalue problem

$$\widehat{L}f = \Gamma f, \tag{14}$$

where $\widehat{L} = -\partial_{xx} + W(x)$, $W(x) = 1 - 2/\left(1 + B^2 x^2\right)$.

Note that this is a Schrödinger-like equation (compare Eq. (14) with the one-dimensional time-independent Schrödinger equation, $-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi = E\psi$). As discussed above, the bound states correspond to the soliton modes.

Let us re-write the Eq. (14) in the following form

$$\widehat{M}f = \Delta f, \tag{15}$$

where $\widehat{M} = -\partial_{xx} + V(x)$, $V(x) = -2/(1 + B^2 x^2)$, and $\Delta = \Gamma - 1$.

This will allow us to apply directly some important theorems [65].

Using results from Ref. [65] we will reformulate here a theorem in a way that will be useful for our study of Eq. (15).

Define the Schrödinger equation $-\partial_{xx} f + V(x)f = \Delta f$ and suppose that $V(x)$ is bounded from below, and that $V(x) \to 0$ as $|x| \to \infty$.

The number of bound states is infinite if there exists an $x_* > 0$ such that

$$x^2 V(x) < -1/4, \tag{16}$$

both for $x > x_*$ and for $x < -x_*$.

On the other hand, if there is an $x_* > 0$ such that $x^2 V(x) > -1/4$ for $|x| > |x_*|$, then, the number of bound states is finite.

Let us apply this theorem to our Eq. (15).

We choose $x_* = \sqrt{1/(8 - B^2)}$, where $B^2 < 8$.

As $-2x^2/(1 + B^2 x^2) < -1/4$ for $x > x_*$ and for $x < -x_*$, then we have proved that we can find an x_* such that $V(x) = -2/(1 + B^2 x^2)$ satisfies the conditions (16).

For $B^2 > 8$, the number of bound states is always finite.

The remarkable result is that if $B^2 < 8$, then the perturbed sine-Gordon equation (12) can possess an arbitrarily large number of internal (shape) modes!

5. Conclusions

We have investigated the inhomogeneous sine-Gordon equation given by Eq. (1).

We have found exact kink solutions to the perturbed sine-Gordon equation, and we have been able to study analytically the kink stability problem.

Our conclusion is that a kink equilibrated by an exponentially-localized perturbation has a finite number of oscillation modes, whereas a sufficiently broad equilibrating perturbation supports an infinite number of soliton internal (shape) modes. This phenomenon is particularly relevant in systems with inhomogeneous long-range forces.

We believe that the soliton-like structures described in this paper, which essentially are extended objects with internal (shape) mode oscillations, can play an important role in both Particle Physics and Condensed Matter Theory.

Acknowledgements

References

[1] F.J. Buijnsters, A. Fasolino, M.I. Katsnelson, Phys. Rev. Lett. 113 (2014) 217202.
[2] E. Kim, F. Li, C. Chong, G. Theocharis, J. Yang, P.G. Kevrekidis, Phys. Rev. Lett. 114 (2015) 118002.
[3] D. Backes, F. Macià, S. Bonetti, R. Kukreja, H. Ohldag, A.D. Kent, Phys. Rev. Lett. 115 (2015) 127205.
[4] A. Costa, A.R. Osborne, D.T. Resio, S. Alessio, E. Chrivì, E. Saggese, K. Bellomo, C.E. Long, Phys. Rev. Lett. 113 (2014) 108501.
[5] T.S. Mendonça, H.P. de Oliveira, J. High Energy Phys. 6 (2015) 133.
[6] O. Braun, Y.S. Kivshar, The Frenkel–Kontorova Model: Concepts, Methods, and Applications, Springer Science & Business Media, Berlin, 2013.
[7] A.R. Bishop, J.A. Krumhansl, S.E. Trullinger, Physica D 1 (1980) 1.
[8] Y.S. Kivshar, B.A. Malomed, Rev. Mod. Phys. 61 (1989) 763.
[9] D.W. McLaughlin, A.C. Scott, Phys. Rev. A 18 (1978) 1652.
[10] M. Peyrard, D.K. Campbell, Physica D 9 (1983) 33.
[11] D.K. Campbell, J.F. Schonfeld, C.A. Wingate, Physica D 9 (1983) 1.
[12] D.K. Campbell, M. Peyrard, P. Sodano, Physica D 19 (1986) 165.
[13] Y.S. Kivshar, F. Zhang, L. Vázquez, Phys. Rev. Lett. 67 (1991) 1177.
[14] Y.S. Kivshar, D.E. Pelinovsky, T. Cretegny, M. Peyrard, Phys. Rev. Lett. 80 (1998) 5032.
[15] M.J. Rice, Phys. Rev. B 28 (1983) 3587.
[16] R. Boesch, C.R. Willis, Phys. Rev. B 42 (1990) 2290.
[17] P. Tchofo Dinda, R. Boesch, E. Coquet, C.R. Willis, Phys. Rev. B 46 (1992) 3311.
[18] F. Zhang, V.V. Konotop, M. Peyrard, L. Vázquez, Phys. Rev. E 48 (1993) 548.
[19] F. Zhang, Phys. Rev. E 54 (1996) 4325.
[20] F. Zhang, Physica D 110 (1997) 51.
[21] E. Majerníková, Y.B. Gaididei, O.M. Braun, Phys. Rev. E 52 (1995) 1241.
[22] N.R. Quintero, A. Sánchez, F.G. Mertens, Phys. Rev. E 62 (2000) R60(R).
[23] J.A. González, A. Bellorín, L.E. Guerrero, Phys. Rev. E 65 (2002) 065601(R).
[24] G. Kälbermann, J. Phys. A, Math. Gen. 37 (2004) 11603.
[25] I.V. Barashenkov, O.F. Oxtoby, Phys. Rev. E 80 (2009) 026608.
[26] O.F. Oxtoby, I.V. Barashenkov, Phys. Rev. E 80 (2009) 026609.
[27] S. Cuenda, N.R. Quintero, A. Sánchez, Discrete Contin. Dyn. Syst., Ser. S 4 (2011) 1047.
[28] J.A. González, J.A. Hołyst, Phys. Rev. B 45 (1992) 10338.
[29] J.A. González, B.A. Mello, L.I. Reyes, L.E. Guerrero, Phys. Rev. Lett. 80 (1998) 1361.
[30] J.A. González, L.E. Guerrero, A. Bellorín, Phys. Rev. E 60 (1999) R37.
[31] J.A. González, S. Cuenda, A. Sánchez, Phys. Rev. E 75 (2007) 036611.
[32] J.A. González, M.A. García-Ñustes, A. Sánchez, P.V.E. McClintock, New J. Phys. 10 (2008) 113015.
[33] M.A. García-Ñustes, J.A. González, Phys. Rev. E 86 (2012) 066602.
[34] D.R. Gulevich, F.V. Kusmartsev, S. Savel'ev, V.A. Yampol'skii, F. Nori, Phys. Rev. B 80 (2009) 094509.
[35] A.M. Gumerov, E.G. Ekomasov, F.K. Zakir'yanov, R.V. Kudryavtsev, Comput. Math. Math. Phys. 54 (2014) 491.
[36] L.W. Dong, F.W. Ye, J.D. Wang, T. Cai, Y.-P. Li, Physica D 194 (2004) 219.
[37] B.Z. Zhang, H.C. Wang, W.L. She, Chin. Phys. 16 (2007) 1052.
[38] J. Liang, B. Xu, Z. Cai, J. Opt. Soc. Am. B 28 (2011) 93.
[39] Z. Cai, J. Liang, L. Yi, L. Deng, Opt. Commun. 283 (2010) 3342.
[40] P. Liang, B. Xu, J. Liang, Z. Cai, Opt. Commun. 284 (2011) 1859.
[41] Z. Cai, B. Xu, W. Gong, Y. Xu, J. Liang, Optik 124 (2013) 1029.
[42] A.M. Gumerov, E.G. Ekomasov, R.R. Murtazin, V.N. Nazarov, Comput. Math. Math. Phys. 55 (2015) 628.
[43] E.G. Ekomasov, A.M. Gumerov, R.R. Murtazin, Math. Methods Appl. Sci. (2016), http://dx.doi.org/10.1002/mma.3908.
[44] L. Morales-Molina, N.R. Quintero, F.G. Mertens, A. Sánchez, Phys. Rev. Lett. 91 (2003) 234102.
[45] N.R. Quintero, A. Sánchez, F.G. Mertens, Phys. Rev. Lett. 84 (2000) 871.
[46] J.A. González, A. Bellorín, L.E. Guerrero, Phys. Lett. A 338 (2005) 60.
[47] R. Chacón, A. Bellorín, L.E. Guerrero, J.A. González, Phys. Rev. E 77 (2008) 046212.
[48] T.J. Goldberg, Regular and Chaotic Dynamics of the Resonantly Driven Wobbling Kink, Ph.D. Thesis, University of Cape Town, 2012.
[49] Y. Jiang, S.-H. Dong, M. Lozada-Cassou, Phys. Rev. E 70 (2004) 026214.
[50] F. Sagués, J.M. Sancho, J. García-Ojalvo, Rev. Mod. Phys. 79 (2007) 829.
[51] C.-H. Wang, T.-M. Hong, R.-K. Lee, D.-W. Wang, Opt. Express 20 (2012) 22675.
[52] H. Li, S.Q. Shen, J.Q. Liang, W.M. Liu, Phys. Rev. B 72 (2005) 014546.
[53] F.K. Abdullaev, J. Garnier, Phys. Rev. A 75 (2007) 033603.
[54] D. Valenti, C. Guarcello, B. Spagnolo, Phys. Rev. B 89 (2014) 214510.
[55] G.P. de Brito, R.A.C. Correa, A. de Souza Dutra, Phys. Rev. D 89 (2014) 065039.
[56] T.S. Mendonça, H.P. de Oliveira, J. High Energy Phys. 09 (2015) 120.
[57] A.E. Bernardini, M. Chinaglia, R. da Rocha, Eur. Phys. J. Plus 130 (2015) 97.
[58] J. Goldstone, R. Jackiw, Phys. Rev. D 11 (1975) 1486.

[59] J.F. Currie, J.A. Krumhansl, A.R. Bishop, S.E. Trullinger, Phys. Rev. B 22 (1980) 477.

[60] S. Jeyadev, J.R. Schrieffer, Synth. Met. 9 (1984) 451.

[61] M.B. Fogel, S.E. Trullinger, A.R. Bishop, J.A. Krumhansl, Phys. Rev. Lett. 36 (1976) 1411.

[62] M.B. Fogel, S.E. Trullinger, A.R. Bishop, J.A. Krumhansl, Phys. Rev. B 15 (1977) 1578.

[63] J.A. González, Mod. Phys. Lett. B 6 (1992) 1867.

[64] S. Flügge, Practical Quantum Mechanics, Springer, Berlin, 1971.

[65] K. Chadan, N.N. Khuri, A. Martin, Tai Tsun Wu, J. Math. Phys. 44 (2003) 406.