Pergamon

Chaos, Solitons & Fractals Vol. 6, pp. 151–155, 1995

0960-0779/95 $9.50 + .00

0960-0779(94)00257-6

# Noise-Induced Organization in a Sine-Gordon Chain

L. E. GUERRERO and R. RANGEL

Centro de Física, Instituto Venezolano de Investigaciones Científicas
Apartado 21827, Caracas 1020 A, Venezuela and
Departamento de Física, Universidad Simón Bolívar
Apartado 89000, Caracas 1080 A, Venezuela

**Abstract** - The appearance of thermally activated solitons in a sine-Gordon system is related to the roughening exponent $\zeta$, defined as the scaling exponent of the length of the ensemble average of the standard deviation of the height of the spatiotemporal profile. Before the onset of the noise-induced transition to the solitonic regime, the roughening exponent is zero as it is for a white noise signal. After the activation of solitons, there is a very interesting crossover from non-KPZ behavior ($\zeta \sim 0.70$) to KPZ behavior ($\zeta \sim 0.50$); additionally, for sufficiently large scales, a crossover to a zero roughening exponent takes place. In this paper we precise the underlying dynamics of the different regimes that appear at different scales via geometric characterization as the size of the system and the friction coefficient are varied.

## INTRODUCTION

Nonequilibrium processes can give rise to interesting self-similar and universal behavior [1]. However, the mechanisms of self-organization and emergence of cooperative behavior are not well understood.

The driven and damped sine-Gordon equation displays many of the phenomena commonly associated with complex systems, particularly, it exhibits noise-induced pattern formation. The randomly driven sine-Gordon system experiences a sharp crossover to an ordered regime characterized by a low number of degrees of freedom and associated with the activation of solitonic excitations [2]. After the onset of the solitonic regime the spatiotemporal profiles exhibit two different self-affined regimes [3]. One of these regimes can be related with the Kardar-Parisi-Zhang equation (KPZ) or random Burgers equation, which has been very successful in explaining a broad range of structures generated by non-equilibrium stochastic processes [4]. In this paper we precise the underlying dynamics by geometrically characterizing the different regimes that appear at different scales as the relevant parameters are varied.

## THE RANDOM SINE-GORDON CHAIN

The sine-Gordon equation $\phi_{xx} - \phi_{tt} - \sin\phi = 0$ models a chain of coupled pendula. We couple this chain of pendula to a heat bath following the Langevin approach, obtaining in this way the random and damped sine-

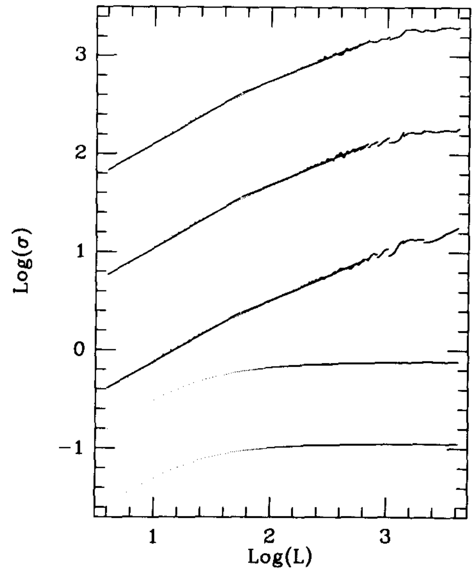


Fig. 1. *log* $\sigma$ vs. *log* $L$ for increasing values of the variance of the noise. We present for the curves corresponding to solitonic regimes linear fits with slopes ~ $0.70$ and ~ $0.50$. We discretized the equation into *4096* points and the parameters values are $\alpha = 0.252$ and $l = 160$.

Gordon model:

$$\phi_{xx} - \phi_{tt} - \sin\phi = \alpha\phi_t - R(x,t) \qquad (1)$$

All equations are written in their dimensionless form with all quantities measured in their natural units. The first term on the right-hand side of Eq. (1) is the loss term representing the energy flow to the reservoir, while the second term is the noise associated with $\alpha$, giving the disordered thermal-energy input to the system. The noise term is uncorrelated both in time and space. The effect of the random force $R(x,t)$ is "to heat" the pendula: a soliton-like excitation can appear when a given pendulum escapes from its potential well. We use flat initial conditions ($\phi(x,0) = \phi_t(x,0) = 0$) and open boundary conditions ($\phi_x(0,t) = \phi_x(L,t) = 0$). The parameters of our simulations $\Delta x = 0.039$ and $\Delta t = 0.035$.

The random and damped sine-Gordon model describes nucleation-dominated crystal growth [5] if one considers the solution $\phi(x,t)$ as the height of a one dimensional surface.

## THE SOLITON GAS AS A RANDOM FRACTAL

In Fig. 1 ($\alpha = 0.252$) we present a log-log plot of $\sigma(L)$ as a function of $L$, where $\sigma(L)$ is the ensemble average of the standard deviation of the spatiotemporal profile as a function of the length scale $L$, i.e.,

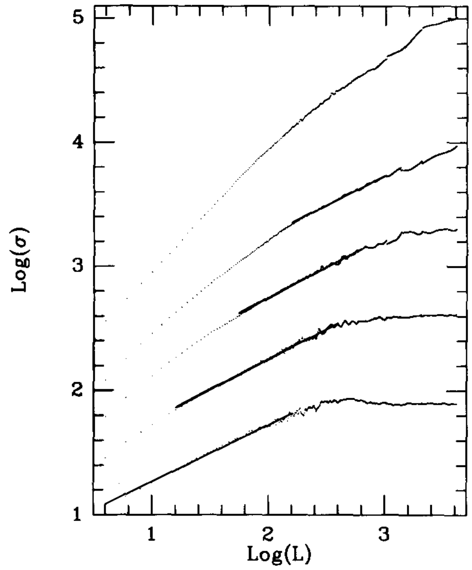


Fig. 2. *log* $\sigma$ vs. *log* $L$ for different values of the measure of the damping $\alpha$ (from below $\alpha$ equals $25.2$, $2.52$, $0.252$, $0.0252$ and $0.0$) and $l = 160$.

$$\sigma(L) = \left\langle \left[ \frac{1}{N_L} \sum_{i-1}^{N_L} (\phi_i - \bar{\phi})^2 \right]^{1/2} \right\rangle ; \bar{\phi} = \frac{1}{N_L} \sum_{i-1}^{N_L} \phi_i \tag{2}$$

Here $\langle ... \rangle$ means ensemble average. In our simulations we average over 5000 realizations for each size $L$ sampled once the stationary regime has settled down, in such a way that $\sigma(t, L) = \sigma(L)$. We obtain different ensembles by taking the surface $\phi(x,t)$ separated by a long enough time interval and by dividing the total length $l = 160$ in segments of size $L$. In Eq. (2) the index $i$ runs over the number of points $N_L$ contained in the segment of length $L$.

The surface $\phi(x,t)$ is a random or stochastic function being the solution of a nonlinear evolution equation with noise; if $\phi(x,t)$ is a self-affined function we expect $\sigma(L)$ to scale as $\sigma(L) \propto L^{\zeta}$ [6], where $\zeta$ is the Hurst exponent [7], which is known in the case of surfaces the roughening exponent. This is confirmed in Fig. 1 that shows the expected scaling behavior for five different variances of the noise (from below $1.7$, $10.0$, $20.0$, $283.3$ and $3{,}333.33$). The third curve is around the dynamic transition to the ordered state with a reduced number of degrees of freedom [2].

The roughening exponent $\zeta$ displays beneath the transition (the pair of lower curves) the behavior characteristic of white noise, i.e., $\zeta = 0$. Above the onset of solitons (upper curves) the roughening exponent exhibits two different scaling behaviors revealing a crossover from non-KPZ behavior ($\zeta \sim 0.70$) for small length scales to KPZ behavior ($\zeta \sim 0.50$) [4]; note that the crossover length appears to be the same.

It can be appreciated in the top two curves of Fig. 1 that for sufficiently large scales, the coherence ceases and a crossover to a zero roughening exponent takes place (see top two curves in Fig. 1).

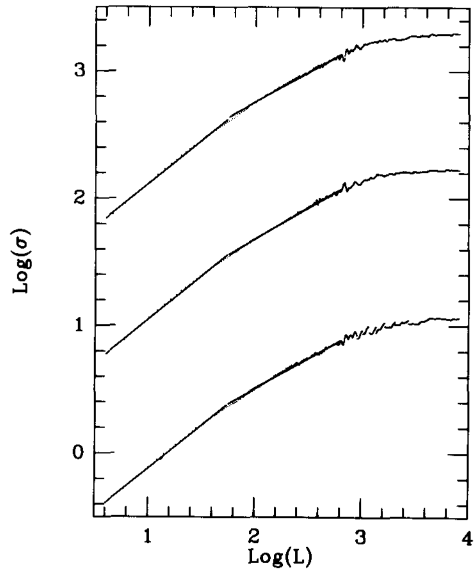


Fig. 3. *log* $\sigma$ vs. *log* $L$ for $\alpha$ = 0.252 and $l$ = 320.

Figure 2 presents log-log plots of $\sigma(L)$ as a function of $L$ for different values of the measure of the damping $\alpha$ (starting from the bottom $\alpha$ equals *25.2, 2.52, 0.252, 0.0252* and *0.0*) while the previous size of the system is preserved and the variance of the noise is *3,333.33*. The lower four cases exhibit good linear fits $\zeta \sim 0.50$ (shown in the Fig. 2). This KPZ-like behavior shifts to intermediate scales of length as the damping is decreased, this is, as the solitonic character of the system strengthens. The lower curve corresponds to the maximum damping and weaker solitonic character of the model; it exhibits only two roughening exponents $\zeta$, *0.5* and zero. The KPZ equation is not a solitonic equation and this is in agreement with the behavior of the random sine-Gordon equation for large damping. As the damping is decreased, the system begins to depart from the KPZ-like behavior at small scales of length; the second and the third curves from below exhibits good fits $\zeta \sim 0.70$ (not shown in the Fig. 2) in this range. The upper curve of Fig. 2 corresponds to a random sine-Gordon equation without damping (so, this case departs from the Langevin approach); this curve exhibits a linear fit $\zeta \sim 1.0$ up to $\log L = 2$ and $\zeta \sim 0.8$ in the range between $\log L = 2$ and $\log L = 2.5$.

In Figure 3 we present log-log plots of $\sigma(L)$ as a function of $L$ for the same parameters of the top three curves of Fig. 1 but the size of the system which has been doubled (as well as the number of discretization points). The crossover length between the non-KPZ behavior ($\zeta \sim 0.70$) for small length scales and the KPZ behavior ($\zeta \sim 0.50$) is preserved when the size of the system is increased. By comparison of the lower curve in Fig. 1 and Fig, 2, we can appreciate that the $\zeta \sim 0.0$ plateau is more sharply defined when the size of the system is increased.

## CONCLUSIONS

Summarizing, we have shown quantitatively that the coherence of the ordered state appears to have three different regimes with well-defined crossover lengths. These spatiotemporal regimes ($\zeta \sim 0.7$, $0.5$ and $0.0$) appear to have different coherent lengths (soliton-like, KPZ-like and none, respectively).

*Acknowledgement-* This work has been partially supported by Consejo Nacional de Investigaciones Científicas y Tecnológicas under project S1-2708. We are grateful to Anwar Hasmy for his helpful assistance in the early stages of this work.

## REFERENCES

1. K. Elder, Langevin simulations of nonequilibrium phenomena, *Computers in Physics* **7**, 27-33 (1993).
2. L. E. Guerrero, A. Hasmy and G. J. Mata, Stochastically-driven coherence in a sine-Gordon chain, *Physica B* **194-196**, 1631-1632 (1994).
3. R. E. Rangel, L. E. Guerrero and A. Hasmy, Roughening transition in a thermal sine-Gordon system, *Physica B* **194-196**, 411-412 (1994).
4. M. Kardar, G. Parisi and Y.-C. Zhang, Dynamic Scaling of growing interfaces, *Phys. Rev. Lett.* **56**, 889-892 (1986); E. Medina, T. Hwa, M. Kardar and Y.-C. Zhang, Burgers equation with correlated noise: Renormalization-group analysis and applications to directed polymers and interface growth, *Phys. Rev. A* **39**, 3053-3075 (1989).
5. J. Krug. and H. Spohn, Anomalous fluctuations in the driven and damped sine-Gordon chain, *Europhys. Lett.* **8**, 219- 224 (1989).
6. B. Mandelbrot, Self-affine fractals and fractal dimension, *Physica Scripta* **32**, 257-260 (1985).
7. F. Family and V. Vicsek (eds.), *Dynamics of Fractal Surfaces*. World Scientific, Singapore (1991).