Response to "Comment on 'Exact solutions to chaotic and stochastic systems'" [Chaos 13, 123 (2003)]

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In the previous Comment[1] James says that truly random numbers are by definition produced by a physical process.

We agree that this can be a definition. However, we should note that there are other definitions of randomness. For instance, there is the algorithmic complexity theory.[2–6] In this theory the complexity of a sequence is given by the length of the smallest program (measured in bits) necessary for the computation of the sequence. The shortest program which produces a random sequence cannot be compressed and its length is of the order of the length of the sequence itself. The algorithmic complexity theory is a very beautiful theory. Many strong theorems have been proved in the framework of this theory. However, the algorithmic complexity theory cannot be applied to the investigation of an empirical, experimental time series because, in general, it is impossible to determine the length of the program that produced the time series. For example, the digits of some computable irrational numbers look perfectly random to a statistician, but the required computer programs are very short. In fact, randomness in a statistical sense and randomness based on the program length are very different. The reason for this difference is that it is possible to write a program which gives the first N digits of some computable irrational number, but it is very difficult to write a program which gives N consecutive digits starting at some random position.[7] Besides, the results of the algorithmic complexity theory are not applicable to the actual generation of random sequences.[8–12] There are computability problems in the algorithmic complexity theory.[8–12]

Recently, there have been very important developments in the study of randomness based on the concept of unpredictability.[8–18] The concept of unpredictability is related to the intuitive notion that random sequences should be unpredictable in advance. On the other hand, the concept of unpredictability can be shown to be related to the concept of statistically independent numbers. In particular, we believe that this should be the most important property of random sequences.

To be precise, we will consider a sequence unpredictable if given any finite string $X_0, X_1, X_2, \ldots, X_m$ (where m can be any integer), anytime we find this string again in the sequence in several places, the next number X_{m+1} can take different values. That is, the next value is not determined by the previous values.

Now let us make the following "thought experiment." Suppose we have found some mathematical method for producing unpredictable time series. These numerical time series are transformed into analog signals using a converter. Then these analog signals are used as the voltage source in some experiment with nonlinear circuits. After that, the current in some point of the circuit is measured by a physicist, who does not know the general mechanism of the circuit and the source of the noise. For him this is a black box. Anyway he is sure that this is a physical process. From the observations he cannot obtain a law that could describe this time series. For him this is like noise. Is this a random process?

Recently[19,20] we have shown that functions of type $X_n = P(\theta z^n)$ (where $P(t)$ is a periodic function and z is a noninteger number) can produce random dynamics. Using the theory of these functions we have shown that autonomous dynamical systems can also generate random dynamics. The following system is an example:

$$X_{n+1} = \begin{cases} aX_n, & \text{if } X_n < Q, \\ bY_n, & \text{if } X_n > Q, \end{cases} \tag{1}$$

$$Y_{n+1} = cZ_n, \tag{2}$$

$$Z_{n+1} = \sin^2(\pi X_n). \tag{3}$$

Moreover, with the help of this theory we have been able to predict new random phenomena. These predictions have been corroborated by real experiments.

In Ref. 20 we present real physical systems that can produce this kind of random time series. We have performed real experiments with some of these systems. In particular, we have shown that a Josephson junction coupled to a chaotic circuit can generate random dynamics. We have applied several statistical methods to the produced time series. The general result is that the dynamics behaves as random noise. For these systems we have mathematical models. The models produce the same kind of random dynamics generated by the physical systems.

This is a random dynamics produced by a physical process. So it is truly random. Can we call the same dynamics generated by the mathematical model truly random?

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